Mathematical Physics
Group Contest

June 11, 2023

You can choose 2 out of the following 3 questions.

Question 1. Consider a free electron-positron system with linearly spaced energy level $E_n = \epsilon_0 \left(n - \frac{1}{2}\right)$ for $n \in \mathbb{Z}$, and total fermion number $N = N_{\mathrm{e}} - N_{\mathrm{p}}$. Define $q = e^{-\epsilon_0/T}$ and $w = e^{\mu/T}$ for the energy and fermion number fugacities, then the grand canonical partition function is given by

$$Z(w,q) = \sum_{\substack{\text{fermion}\\\text{occupations}}} e^{-E/T+\mu N/T} = \sum_{N=-\infty}^{+\infty} w^N Z_N(q)\,,$$

where $Z_N(q)$ is the partition function counting states for given fermionic number N.

(1) From the perspective of "Dirac sea" (The vacuum is the state in which all negative energy states are filled. Therefore an electron is a state created above the vacuum, while a positron is a "hole" state where all negative energy states are occupied except one), show that

$$Z_0(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}\,,$$

and thus

$$Z_N(q) = q^{\frac{N^2}{2}} Z_0(q)\,;$$

(2) In a modern QFT perspective the electron and positron states are produced by two kinds of anti-commuting creation operators acting on the vacuum. Rewrite the partition function $Z(w,q)$ in this way, and thus prove the Jacobi's triple product identity

$$\prod_{n=1}^{\infty} (1-q^n)\left(1 + w\,q^{n-\frac{1}{2}}\right)\left(1 + w^{-1} q^{n-\frac{1}{2}}\right) = \sum_{N=-\infty}^{+\infty} w^N q^{\frac{N^2}{2}}\,.$$

(3) Can you name a relativistic model with this partition function?

Question 2. Consider the following one-dimensional Hamiltonian defined on the infinite line $-\infty < x < \infty$ with a well-like potential $V(x)$:

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + V(x), \qquad V(x) = \begin{cases} -V_0/L, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$$

where $V_0 > 0$. We are interested in the bound states of $\hat{H}$, i.e. eigenstates of $\hat{H}$ with negative eigenvalues.

(1) Compare the two limits $L \to 0$ and $L \to \infty$: which limit has the largest number of bound states?

(2) What is the minimum number of bound states as we vary $L \in (0, \infty)$?

(3) Modify the potential $V(x)$ to the following form:

$$V(x) = \begin{cases} 0, & x > L/2 \\ -V_0/L, & -L/2 < x < L/2 \\ +\infty, & x < -L/2 \end{cases}$$

What is the minimum number of bound states as we vary $L \in (0, \infty)$?

Question 3. Consider the QFT of an Abelian gauge field A_μ in 3-dimensional Minkowski space with action

$$S = -\frac{1}{4g_{\mathrm{YM}}^2}\int d^3x\, F_{\mu\nu}F^{\mu\nu} + \lambda \int d^3x\, \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field strength and λ is a real coupling.

Answer the following questions:

(1) is this theory consistent at the full quantum level? Motivate your answer

(2) ***answer to the following questions only if you said "yes" to question (1)***

(2a) which symmetries which are unbroken at $\lambda = 0$ get broken for $\lambda \neq 0$?

(2b) when $\lambda \neq 0$ is the theory gapped?

(2c) does this theory contain gauge-invariant conserved currents J^a_μ (that is, such that $\partial^\mu J^a_\mu = 0$) which allow us to define *non-identically-zero* internal quantum numbers (i.e. charges) $Q^a = \int d^2x\, J^a_0(x)$? ***If yes***, say how many non-trivial Q^a's there are, and which compact Lie group G they generate.

(2d) determine the spectrum of the theory, namely list its particle content (mass, spin, # of states at fixed momentum, and internal quantum numbers Q^a for each particle in the spectrum).